A Central Limit Theorem for Functionals of Gaussian Processes

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I dedicate this thesis to my wife, Maila, and my two daughters Maxyn Flaura and Hally Beatrix. I love you, you are my inspiration.

Contents

Abstract

The aim of this thesis is to study and show, as described in the works of Nualart, that a sequence of functionals of Gaussian processes that belongs to a Wiener chaos of fixed order converges in distribution to a standard normal law. First, we will prove this in the finite-dimensional case and then extend this to the infinite-dimensional case. As an example, we will illustrate the classical Central Limit Theorem. We will also show how to apply our result to Gaussian Moving Averages.

Chapter 1

Introduction

The Central Limit Theorem enjoys a very unique position in mathematics especially in the realm of probability and statistics. It is also known as the second fundamental theorem of probability. Together with the Law of Large numbers, which is the first, they constitute the two most important results in probability theory.

Sir Francis Galton, a statistician and biometrician who lived in 19th century England, was quoted to have said the following about the celebrated central limit theorem and about the normal or Gaussian distribution:

> I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the ”Law of Frequency of Error”. The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement, amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshaled in the order of their magnitude, an unsuspected and most beautiful form of regularity proves to have been latent all along.”

First we recall what it means for a stochastic process to be a Gaussian process. The following definition is from [4]:

Definition 1. *An $\mathbb{R}^N$-valued random variable $X = (X_1, ..., X_N)$ is Gaussian (or multivariate normal) if every linear combination $\sum_{j=1}^{N} a_j X_j$ has a one-dimensional normal distribution $\gamma = N(\mu, \sigma^2)$.*

In the above definition, we have used the usual notation $N(\mu, \sigma^2)$ to denote the normal distribution with parameters (μ, σ^2) where $\mu \in \mathbb{R}, \sigma^2 \geq 0$ whose density is given by

$$w_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \, -\infty < x < \infty, \tag{1.1}$$

when $\sigma^2 > 0$. In most part of this paper, we set $\mu = 0$ and $\sigma^2 = 1$.

We will now state our Central Limit Theorem: Consider a sequence of random variables $\{F_n\}_{n\geq 1}$ that belongs to the kth Wiener chaos, for some $k \geq 1$ and such that $\mathbb{E}(F_n^2) \to \sigma^2$ as $n \to \infty$. We will show that this sequence converges in distribution to a normal law $N(0, \sigma^2)$ if and only if one of the following conditions holds:

1. $\mathbb{E}({F_n}^4) \longrightarrow 3\sigma^4$ as $n \to \infty$

2. $\|DF_n\|^2 \longrightarrow k\sigma^2$ in L^2 as $n \to \infty$.

We recall the two notions of convergence that are used in our Central Limit Theorem. For $\{X_n\}_{n\geq 1}$, X that are $\mathbb{R}^N$-valued random variables, we say that X_n *converges in distribution* to X (or equivalently, X_n *converges in law* to X) if and only if

$$\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)],$$

for all continuous, bounded, and real-valued functions f on $\mathbb{R}^N$. We say that X_n *converges in* L^2 to X if $|X_n|, |X|$ are in L^2 and

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^2] = 0.$$

The Central Limit Theorem that we will prove in this paper have been used and discussed in the works of Nualart and Peccati [6]. Malliavin Calculus played a major role in their work. Malliavin calculus, named after the French mathematician Paul Malliavin, provides the mechanics to compute derivatives of random variables. In Nualart and Peccati's paper, however, the proof of the Central Limit Theorem is rather complicated and very involved. Here, we will present an elementary proof of the Central Limit Theorem by developing the ideas for the finite-dimensional case and then carefully extending it to the infinite-dimensional case. Examples and applications are provided.

This paper is organized as follows. In Section 2, we introduce some definitions, notations and preliminary results that will be used to provide proofs for the main result. In Section 3, we state and prove the main result in the finite-dimensional case. Section 4 deals with the infinite dimensional case. Finally, in Section 5, we discuss two main examples - the first is the classical Central Limit Theorem while the second is an application involving Gaussian Moving Averages.

Chapter 2

Preliminaries, Definitions and Notations

The first sub-section discusses Hermite polynomials and their basic properties. The important result in this sub-section is the orthogonality property of Hermite polynomials applied to random variables with Gaussian distribution. In the next sub-section, we produce a basis for the linear space $L^2(\Omega)$ via a Gaussian family obtained from Hermite polynomials. We also introduce Wiener chaos H_k of order k. The main result in the second sub-section is that the space $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^N$ can be decomposed into an infinite orthogonal sum of subspaces $\mathscr{H}_k$. The third sub-section introduces two operators and an identity that relates them. These two operators will be the key in proving our Central Limit Theorem in the third section.

2.1 Hermite Polynomials

Let us recall the definition of a special sequence of polynomials called Hermite polynomials.

Definition 2. *For any $k \geq 1$, the kth Hermite polynomial is defined by*

$$h_k(x) = \frac{(-1)^k}{k!} e^{\frac{x^2}{2}} \frac{d^k}{dx^k}(e^{-\frac{x^2}{2}}), \tag{2.1}$$

and $h_0(x) = 1$.

Here are some of the first few Hermite polynomials:

$$\begin{aligned}
k = 0 &\quad : \quad h_0(x) = 1 \\
k = 1 &\quad : \quad h_1(x) = x \\
k = 2 &\quad : \quad h_2(x) = \frac{1}{2}(x^2 - 1) \\
k = 3 &\quad : \quad h_3(x) = \frac{1}{3!}(x^3 - 3x) \\
k = 4 &\quad : \quad h_4(x) = \frac{1}{4!}(x^4 - 6x^2 + 3)
\end{aligned}$$

Some properties of the Hermite polynomials:

- The Hermite polynomials (2.1) can be generated by using the following exponential generating function:

  $$e^{tx - \frac{t^2}{2}} = \sum_{k=0}^{+\infty} t^k h_k(x).$$

  Indeed, for any x and t, define

  $$E(x,t) = \exp(tx - \frac{t^2}{2}). \tag{2.2}$$

  Then

  $$E(x,t) = \exp(tx - \frac{t^2}{2}) = \exp\left[\frac{x^2}{2} - \frac{1}{2}(x-t)^2\right].$$

  Taking the kth partial derivative of E with respect to t, we have

  $$\frac{d^k}{dt^k} E(x,t) = e^{\frac{x^2}{2}} \frac{d^k}{dt^k} e^{-\frac{1}{2}(x-t)^2} = \left[e^{\frac{x^2}{2}} (-1)^k \frac{d^k}{dy^k} e^{-\frac{y^2}{2}}\right]_{y=x-t}$$

For $t = 0$, we get $(-1)^k e^{\frac{x^2}{2}} \dfrac{d^k}{dx^k} e^{-\frac{x^2}{2}} = h_k(x)$.

- A Hermite polynomial h_k is a polynomial of degree k, where $k = 0, 1, 2, \ldots$. The most important property of Hermite polynomials is their orthogonality. They are orthogonal with respect to the weighted L^2 inner product with weight function $w(x) = e^{-\frac{x^2}{2}}$, that is,

$$\int_{-\infty}^{\infty} h_m(x) h_n(x) w(x) dx = 0$$

when $m \neq n$, and

$$\int_{-\infty}^{\infty} h_n(x) h_n(x) w(x) dx = \frac{1}{n!}\sqrt{2\pi}$$

when $m = n$. Here, we are using the inner product

$$< f, g >= \int_{-\infty}^{+\infty} f(x) g(x) w(x)\, dx.$$

These properties are consequences of Theorem 2.2 below.

- Like Jacobi and Laguerre polynomials, Hermite polynomials satisfy

$$h_k'(x) = h_{k-1}(x) \text{ for } k \geq 1, \tag{2.3}$$

that is, they form an Appell sequence. This is a very nice property; notice that the subscript k has been shifted by 1 to the left when one differentiates h_k with respect to x. The key to proving this is that

$$\frac{\partial E(x,t)}{\partial x} = t \exp\left(tx - \frac{t^2}{2}\right) = tE(x,t).$$

For a discussion of this fact, see [10]. This Appell sequence-property of the Hermite polynomials will be used in Section 4.

- An iterative identity that involves the first derivative of Hermite polynomials is

$$(k+1)h_{k+1}(x) = xh_k(x) - h_k'(x), \text{ for } k \geq 1. \tag{2.4}$$

Indeed, using the definition (2.1), we have

$$\begin{aligned}
h_k'(x) &= \frac{(-1)^k}{k!}e^{\frac{x^2}{2}}x\frac{d^k}{dx^k}(e^{-\frac{x^2}{2}}) + \frac{(-1)^k}{k!}e^{\frac{x^2}{2}}\frac{d^{k+1}}{dx^{k+1}}(e^{-\frac{x^2}{2}}) \\
&= \frac{(-1)^k}{k!}e^{\frac{x^2}{2}}\left[x\frac{d^k}{dx^k}(e^{-\frac{x^2}{2}}) + \frac{d^{k+1}}{dx^{k+1}}(e^{-\frac{x^2}{2}})\right] \\
&= \frac{(-1)^k}{k!}e^{\frac{x^2}{2}}\left[x\frac{k!}{(-1)^k}e^{-\frac{x^2}{2}}h_k(x) + \frac{(k+1)!}{(-1)^{k+1}}e^{-\frac{x^2}{2}h_{k+1}}(x)\right] \\
&= xh_k(x) - (k+1)h_{k+1}(x).
\end{aligned}$$

- For any $k \geq 1$ we have

$$h_k''(x) = xh_k'(x) - kh_k(x). \tag{2.5}$$

To see this, the previous result says that $h_k'(x) = xh_k(x) - (k+1)h_{k+1}(x)$ and so by taking the derivative again,

$$\begin{aligned}
h_k''(x) &= h_k(x) + xh_k'(x) - (k+1)h_{k+1}'(x) \\
&= h_k(x) + xh_k'(x) - (k+1)h_k(x) \\
&= h_k(x) + xh_k'(x) - kh_k(x) - h_k(x) \\
h_k''(x) &= xh_k'(x) - kh_k(x).
\end{aligned}$$

This second-derivative-property (2.5) will be used to prove an identity in Section 2.3.

The importance of Hermite polynomials in probability is best captured in our first theorem; this is the most important result for this sub-section: it proves the orthogonality of Hermite polynomials of Gaussian variables. This theorem will be used in the next sub-section to formulate a family generated from the Hermite polynomials. Such a family will form a basis for $L^2(\Omega)$.

Theorem 3. *Let X,Y be two random variables with joint Gaussian distribution such that $\mathbb{E}(X)=\mathbb{E}(Y)=0$ and $\mathbb{E}(X^2)=\mathbb{E}(Y^2)=1$. Then for all $n,m\geq 0$ we have*

$$\mathbb{E}(h_n(X)\cdot h_m(Y))=\begin{cases}0, & n\neq m\\ \dfrac{1}{n!}(\mathbb{E}(XY))^n, & n=m,\end{cases}$$

where $\{h_k\}_{k=0}^{\infty}$ is the sequence of Hermite polynomials.

Proof. This proof comes from [5]. For all $s,t\in\mathbb{R}$ we have

$$\mathbb{E}\left(\exp\left(sX-\frac{s^2}{2}\right)\exp\left(tY-\frac{t^2}{2}\right)\right)=\exp\left(st\mathbb{E}\left(XY\right)\right).$$

Taking the $(n+m)th$ partial derivative $\dfrac{\partial^{n+m}}{\partial s^s\partial t^m}$ at $s=t=0$ in both sides of the above equality yields,

$$\mathbb{E}(n!m!h_n(X)\cdot h_m(Y))=\begin{cases}0, & n\neq m\\ n!(\mathbb{E}(XY))^n, & n=m.\end{cases}$$

□

2.2 Orthogonal Decomposition

Suppose that $\xi = (\xi_1, \cdots, \xi_N)$ is an N-dimensional random vector such that ξ_i are independent $N(0,1)$ random variables defined on the canonical probability space $\Omega = \mathbb{R}^N$, that is, $\xi_i : \mathbb{R}^N \longrightarrow \mathbb{R}$ are given by $\xi_i(x) = x_i$. Consider the family of random variables

$$\mathsf{X}_\alpha = \prod_{i=1}^{N} h_{\alpha_i}(\xi_i), \text{where } \alpha = (\alpha_1, \cdots, \alpha_N), \alpha_i \in \{0,1,2,...\}. \tag{2.6}$$

In the above notation, α is called a multi-index. Let us give specific cases to illustrate this family of random variables X_α.

1. Set $N = 1$. Then $\mathsf{X}_\alpha = \prod_{i=1}^{1} h_{\alpha_1}(\xi_1) = h_{\alpha_1}(\xi_1)$. This is a Hermite polynomial of degree $\alpha_1 \geq 0$. For instance,

$$\begin{aligned}
&\text{if } \alpha_1 = 1 \quad \text{then} \quad \mathsf{X}_{(1)} = h_1(\xi_1) = \xi_1. \\
&\text{if } \alpha_1 = 2 \quad \text{then} \quad \mathsf{X}_{(2)} = h_2(\xi_1) = \frac{1}{2}(\xi_1^2 - 1). \\
&\text{if } \alpha_1 = 3 \quad \text{then} \quad \mathsf{X}_{(3)} = h_3(\xi_1) = \frac{1}{3!}(\xi_1^3 - 3\xi_1).
\end{aligned}$$

   Therefore, when $N = 1, \mathsf{X}_\alpha$ represents the family of Hermite polynomials in a single variable ξ_1 of degree $\alpha_1 \geq 0$.

2. Set $N = 2$. Then $\mathsf{X}_{\alpha=(\alpha_1,\alpha_2)} = \prod_{i=1}^{2} h_{\alpha_i}(\xi_i) = h_{\alpha_1}(\xi_1) \cdot h_{\alpha_2}(\xi_2)$. This generates a product of two Hermite polynomials that will have a degree of $\alpha_1 + \alpha_2$ and

$\alpha_i \geq 0$. For instance, if $\alpha = (\alpha_1, \alpha_2) = (1,2)$, then

$$\begin{aligned} \mathsf{X}_{(\alpha_1,\alpha_2)} = \mathsf{X}_{(1,2)} &= \prod_{i=1}^{2} h_{\alpha_i}(\xi_i) \\ &= h_1(\xi_1) \cdot h_2(\xi_2) \\ &= \xi_1 \cdot \frac{1}{2}(\xi_2^2 - 1) \\ &= \frac{1}{2}(\xi_1 \xi_2^2 - \xi_1) \end{aligned}$$

and $\mathsf{X}_{(\alpha_1,\alpha_2)}$ is a polynomial of degree $\alpha_1 + \alpha_2$.

Or, if $\alpha = (\alpha_1, \alpha_2) = (4,3)$ then,

$$\begin{aligned} \mathsf{X}_{(4,3)} &= \prod_{i=1}^{2} h_{\alpha_i}(\xi_i) \\ &= h_4(\xi_1) \cdot h_3(\xi_2) \\ &= \frac{1}{4!}(\xi_1^4 - 6\xi_1^2 + 3) \cdot \frac{1}{3!}(\xi_2^3 - 3\xi_2) \\ &= \frac{1}{4!}\frac{1}{3!}(\xi_1^4 \xi_2^3 - 3\xi_1^4 \xi_2 - 6\xi_1^2 \xi_2^3 - 18\xi_1^2 \xi_2 + 3\xi_2^3 - 9\xi_2) \end{aligned}$$

3. Set $N = 3$. Then $\alpha = (\alpha_1, \alpha_2, \alpha_3), \alpha_i \geq 0$, and so we have,

$$\mathsf{X}_\alpha = \prod_{i=1}^{3} h_{\alpha_i}(\xi_i) = h_{\alpha_1}(\xi_1) \cdot h_{\alpha_2}(\xi_2) \cdot h_{\alpha_3}(\xi_3)$$

which is a polynomial in three variables ξ_1, ξ_2, ξ_3. A polynomial resulting from the products of three Hermite polynomials that will have a degree equal to $\alpha_1 + \alpha_2 + \alpha_3$.

From these examples, we now see that for a fixed N, the set $\{\mathsf{X}_\alpha : \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N), \alpha_i \geq 0\}$ is a collection of products of Hermite polynomials, the degree of which is the sum

$\sum_{i=1}^{N} \alpha_i$. The number

$$|\alpha| = \sum_{i=1}^{N} \alpha_i$$

is sometimes referred to as the *order* of the multi-index α. Also, we define the *factorial* of the multi-index by

$$\alpha! = \prod_{i=1}^{N} (\alpha_i!).$$

For instance if $N = 2$, then $\alpha = (\alpha_1, \alpha_2), \alpha_i \geq 0, |\alpha| = \alpha_1 + \alpha_2$, and

$$\mathsf{X}_\alpha = \mathsf{X}_{(\alpha_1,\alpha_2)} = \{\mathsf{X}_{(0,0)}, \mathsf{X}_{(0,1)}, \cdots, \mathsf{X}_{(1,0)}, \mathsf{X}_{(1,1)}, \cdots\}.$$

If $N = 3$, then $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha_i \geq 0, |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and

$$\mathsf{X}_\alpha = \mathsf{X}_{(\alpha_1,\alpha_2,\alpha_3)} = \{\mathsf{X}_{(0,0,0)}, \cdots\}.$$

The next proposition is used to prove the main result in this section. The orthogonality of the Hermite polynomials is the key ingredient in the proof:

Proposition 4. *The family $\{\mathsf{X}_\alpha\}_\alpha$ is mutually orthogonal.*

Proof. We need to show that for $\mathsf{X}_\alpha, \mathsf{X}_\beta \in \{\mathsf{X}_\alpha\}_\alpha$, we have $\mathbb{E}(\mathsf{X}_\alpha \mathsf{X}_\beta) = 0$ if $\alpha \neq \beta$. For some α_i, β_i, we have

$$\mathsf{X}_\alpha = \prod_{i=1}^{N} h_{\alpha_i}(\xi_i),\ \mathsf{X}_\beta = \prod_{i=1}^{N} h_{\beta_i}(\xi_i),$$

$$\begin{aligned} \mathbb{E}(\mathsf{X}_\alpha \mathsf{X}_\beta) &= \mathbb{E}(\prod_{i=1}^{N} h_{\alpha_i}(\xi_i) \cdot \prod_{i=1}^{N} h_{\beta_i}(\xi_i)) \\ &= \prod_{i=1}^{N} \mathbb{E}(h_{\alpha_i}(\xi_i) \cdot h_{\beta_i}(\xi_i)), \end{aligned}$$

by independence. Surely there will be at least one index i such that α_i and β_i are not equal and so by applying Theorem 3,we have $\mathbb{E}(h_{\alpha_i}(\xi_i) \cdot h_{\beta_i}(\xi_i)) = 0$ $\square$

The previous proposition shows that we have orthogonality for different multi-indices α and β. For similar indices $\alpha = \beta$, we have the following result, where again, the key is Theorem 3. Also, this theorem will be used heavily to prove a proposition in Section 4.

Proposition 5. *Let* $\mathsf{X}_\alpha \in \{\mathsf{X}_\alpha\}_\alpha$ *for some multi-index* α. *Then*

$$\mathbb{E}(\mathsf{X}_\alpha^2) = \frac{1}{\alpha!}.$$

Proof. For some α_i, β_i, and by independence, we have

$$\begin{aligned}
\mathbb{E}(\mathsf{X}_\alpha{}^2) &= \mathbb{E}(\prod_{i=1}^{N} h_{\alpha_i}(\xi_i) \cdot \prod_{i=1}^{N} h_{\alpha_i}(\xi_i)) \\
&= \prod_{i=1}^{N} \mathbb{E}(h_{\alpha_i}^2(\xi_i)) \\
&= \mathbb{E}(h_{\alpha_1}^2(\xi_1))\mathbb{E}(h_{\alpha_2}^2(\xi_2)) \cdots \mathbb{E}(h_{\alpha_N}^2(\xi_N)) \\
&= \frac{1}{\alpha_1!} \cdot \frac{1}{\alpha_2!} \cdots \frac{1}{\alpha_N!} \\
&= \frac{1}{\alpha!}
\end{aligned}$$

$\square$

The previous two results lead us to the fact that the family of random variables $\{\mathsf{X}_\alpha\}_\alpha$ forms a basis for $\mathsf{L}^2(\Omega)$. A detailed proof is presented in [5]:

Proposition 6. *The family of random variables* $\{\mathsf{X}_\alpha\}_\alpha$ *forms a basis of* $\mathsf{L}^2(\Omega)$.

To show this, one notes that $\{\mathsf{X}_\alpha\}_\alpha$ forms an orthogonal system and hence, we only need to verify that they form a complete orthonormal system. Completeness in this case means that the zero function is the only function in the space that is orthogonal to all functions in the system. This follows from the observation that

$$\mathbb{E}(Y\mathsf{X}_\alpha) = 0 \text{ for every } \alpha$$

implies that $Y \equiv 0$.

Now, from the Gaussian family of random variables (2.6), we now define a Wiener chaos.

Definition 7. *For each integer $k \geq 1$, the kth Wiener chaos $\mathscr{H}_k$ is the linear span of $\{\mathsf{X}_\alpha : |\alpha| = k\}$, where α is a multi-index.*

Let us look at some examples.

1. Set $N = 2$. Then $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_i \geq 0$. For $k = 1$, $\mathscr{H}_1$ is the linear span of $\{\mathsf{X}_\alpha : |\alpha| = \alpha_1 + \alpha_2 = 1\} = \{\mathsf{X}_{(0,1)}, \mathsf{X}_{(1,0)}\}$ where

$$\mathsf{X}_{(0,1)} = \prod_{i=1}^{2} h_{\alpha_i}(\xi_i) = h_0(\xi_1)h_1(\xi_2) = 1 \cdot \xi_2 = \xi_2$$

$$\mathsf{X}_{(1,0)} = \prod_{i=1}^{2} h_{\alpha_i}(\xi_i) = h_1(\xi_1)h_0(\xi_2) = \xi_1 \cdot 1 = \xi_1$$

and the elements of $\mathscr{H}_1$ are linear combinations of ξ_1 and ξ_2 which gives $\mathscr{H}_1 = \{a\xi_1 + b\xi_2, a, b \in \mathbb{R}\}$.

In general, for $N \geq 1$ and $k = 1$,

$$\mathscr{H}_1 = \{a_1\xi_1 + a_2\xi_2 + ... + a_n\xi_n : a_i \in \mathbb{R}\},$$

that is, $\mathscr{H}_1$ is the linear combination of the ξ's.

2. Set $N = 2$. Then $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_i \geq 0$. For $k = 2$, $\mathscr{H}_2$ is the linear span of $\{\mathsf{X}_\alpha : |\alpha| = \alpha_1 + \alpha_2 = 2, \alpha_i \geq 0\}$ with $(\alpha_1, \alpha_2) \in \{(1,1), (0,2), (2,0)\}$ so that,

$$\begin{aligned}
\mathsf{X}_{(1,1)} &= h_1(\xi_1)h_1(\xi_2) = \xi_1\xi_2 \\
\mathsf{X}_{(0,2)} &= h_0(\xi_1)h_2(\xi_2) = 1 \cdot \frac{1}{2}({\xi_2}^2 - 1) = \frac{1}{2}({\xi_2}^2 - 1) \\
\mathsf{X}_{(2,0)} &= \frac{1}{2}({\xi_1}^2 - 1)
\end{aligned}$$

then we have the spanning set $\{\mathsf{X}_\alpha : \xi_1\xi_2, {\xi_1}^2 - 1, {\xi_2}^2 - 1\}$ Hence, for $N = 2$ and $k = 2$, we have $\mathscr{H}_2 = \{a\xi_1\xi_2 + b({\xi_1}^2 - 1) + c({\xi_2}^2 - 1), a, b, c \in \mathbb{R}\}$

3. Set $N = 3$. Then $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For $k = 2$, $\mathscr{H}_2$ is the linear span of $\{\mathsf{X}_\alpha : |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = 2, \alpha_i \geq 0\}$
and $\alpha \in \{(1,1,0), (0,1,1), (1,0,1), (2,0,0), (0,2,0), (0,0,2)\}$
so we have,

$$\begin{aligned}
\mathsf{X}_{(1,1,0)} &= h_1(\xi_1)h_1(\xi_2)h_0(\xi_3) = \xi_1\xi_2 1 \\
\mathsf{X}_{(0,1,1)} &= \xi_2\xi_3 \\
\mathsf{X}_{(1,0,1)} &= \xi_1\xi_3 \\
\mathsf{X}_{(2,0,0)} &= \frac{1}{2}({\xi_1}^2 - 1) \\
\mathsf{X}_{(0,2,0)} &= \frac{1}{2}({\xi_2}^2 - 1) \\
\mathsf{X}_{(0,0,2)} &= \frac{1}{2}({\xi_3}^2 - 1)
\end{aligned}$$

So when $k = 2$ and $N = 3$, $\mathscr{H}_2 = \{a_1\xi_1\xi_2 + a_2\xi_2\xi_3 + a_3\xi_1\xi_3 + a_4({\xi_1}^2 - 1) + a_5({\xi_2}^2 - 1) + a_6({\xi_3}^2 - 1), a_i \in \mathbb{R}\}$

From Proposition 2.5 follows the orthogonal decomposition of $L^2(\Omega)$ into the sum of the spaces $\mathscr{H}_k$, $k = 0, 1, 2 \ldots$. A formal proof of this important theorem for this sub-section can be obtained from [5].

Theorem 8. *The space $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the subspaces $\mathscr{H}_n$, where $n = 0, 1, 2, \ldots$. We write this as*

$$L^2(\Omega) = \bigoplus_{k=0}^{\infty} \mathscr{H}_k, \text{ where } k = 0, 1, 2, \cdots$$

where $\mathscr{H}_0 = \mathbb{R}$.

2.3 Two Operators and An Identity

In this sub-section, we discuss some operators from Malliavin Calculus that are usually denoted by D and δ. The first is the derivative operator while the second is its adjoint. They are related by the identity $\delta DF(\xi) = kF(\xi)$ on $\mathscr{H}_k$, which we will prove in this section. For more information about Malliavin calculus, the reader is referred to [5]. We consider functions that are in some class.

Definition 9. *A function $f : \mathbb{R}^N \to \mathbb{R}^k$ is a function with polynomial growth if it satisfies*

$$|f(x)| \leq c(1 + |x|^m), \; x \in \mathbb{R}^N$$

for some $c, m > 0$. The function f is said to be C^1 if all of its partial derivatives are continuous. For a C^1-function $f : \mathbb{R}^N \to \mathbb{R}^N$, the divergence of f, denoted by $div(f)$, is *defined by*

$$div(f)(x) = \frac{\partial f_1}{\partial x_1} + ... + \frac{\partial f_N}{\partial x_N}.$$

The first important result for this subsection is

Proposition 10. *If $F : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $u : \mathbb{R}^N \to \mathbb{R}^N$ are C^1-functions on $\mathbb{R}^N$ with polynomial growth, then*

$$\mathbb{E}(\langle DF(\xi), u(\xi)\rangle) = \mathbb{E}(F(\xi)(\delta u)(\xi)) \tag{2.7}$$

where D is the derivative operator

$$DF = \left(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_N}\right)$$

and δ is operator defined by

$$(\delta u)(\xi) = \delta\,(u(\xi)) := \langle u(\xi), \xi\rangle - \; div\; u(\xi). \tag{2.8}$$

The inner-product used in the definition of δ is the inner-product in $\mathbb{R}^N$ so that for an N-dimensional random vector $\xi = (\xi_1, ..., \xi_N)$, we have

$$\delta(u(\xi)) = \sum_{i=1}^{N} u_i(\xi)\xi_i - \sum_{i=1}^{N} \frac{\partial u_i(\xi)}{\partial x_i}$$

Proof. The proof uses integration by parts. Let us first develop the ideas of the proof by looking at the simplest case $N = 1$. Consider functions $F, u : \mathbb{R} \to \mathbb{R}$. Thus, $DF(x) = F'(x), \text{div}(u(x)) = u'(x)$ and so $(\delta u)(x) = xu(x) - u'(x)$. Hence, in this case, we need to prove that

$$\mathbb{E}\left(\langle F'(\xi), (\delta u)(\xi)\rangle\right) = \mathbb{E}\left(F(\xi)(\delta u)(\xi)\right).$$

To verify this, we compute each side independently. The right-hand side is

$$\begin{aligned}
\mathbb{E}\left(F(\xi)(\delta u)(\xi)\right) &= \int_{-\infty}^{\infty} F(x)(\delta u)(x)w(x)dx \\
&= \int_{-\infty}^{\infty} F(x)\left[xu(x)-u'(x)\right]w(x)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\left[xu(x)-u'(x)\right]e^{-\frac{x^2}{2}}dx,
\end{aligned}$$

where $w(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. For the left-hand side, we apply integration by parts, using $r=\frac{1}{\sqrt{2\pi}}u(x)e^{-\frac{x^2}{2}}$, $ds=F'(x)dx$ so that

$$dr=\frac{1}{\sqrt{2\pi}}\left[u(x)e^{-\frac{x^2}{2}}(-x)+e^{-\frac{x^2}{2}}u'(x)\right],\ s=F(x).$$

The left-hand side is

$$\begin{aligned}
\mathbb{E}\left(\langle DF(\xi),u(\xi)\rangle\right) &= \int_{-\infty}^{\infty} F'(x)u(x)w(x)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F'(x)u(x)e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{\sqrt{2\pi}}u(x)F(x)e^{-\frac{x^2}{2}}\Big|_{-\infty}^{+\infty} \\
&- \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\left[-xu(x)e^{-\frac{x^2}{2}}+u'(x)e^{-\frac{x^2}{2}}\right]dx \\
&= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)e^{-\frac{x^2}{2}}\left[-xu(x)+u'(x)\right]dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)e^{-\frac{x^2}{2}}\left[xu(x)-u'(x)\right]dx \\
&= \mathbb{E}\left(F(\xi)(\delta u)(\xi)\right)
\end{aligned}$$

Note that we have used the fact that F and u have polynomial growth so that only one term remains after applying the integration by parts formula. This finishes the verification for the case $N=1$. Now, let $N>1$. Write $u(\xi)=(u_1(\xi),u_2(\xi),...,u_N(\xi))$.

In this case, we have to apply the n-dimensional version of $\mathbb{E}$ carefully and we also apply the integration by parts formula:

$$\begin{aligned}
\mathbb{E}(\langle DF(\xi),u(\xi)\rangle) &= \mathbb{E}(\sum_{i=1}^{N}\frac{\partial F}{\partial \xi_i}u_i(\xi)) \\
&= \sum_{i=1}^{N}\mathbb{E}(\frac{\partial F}{\partial x_i}u_i(\xi)) \\
&= \sum_{i=1}^{N}\int_{\mathbb{R}^N}(2\pi)^{-\frac{N}{2}}e^{-\frac{|x|^2}{2}}\frac{\partial F}{\partial x_i}u_i(x)dx \\
&= \sum_{i=1}^{N}\int_{\mathbb{R}^N}(2\pi)^{-\frac{N}{2}}e^{-\frac{|x|^2}{2}}F_i'(x)u_i(x)dx \\
&= \sum_{i=1}^{N}\int_{\mathbb{R}^N}(2\pi)^{-\frac{N}{2}}e^{-\frac{|x|^2}{2}}F_i(x)(xu_i(x)-u_i'(x))dx \\
&= \mathbb{E}(F(\xi)(\delta u)(\xi)).
\end{aligned}$$

again, we have used the fact that both F,u have polynomial growths so that they are killed when they are multiplied by $e^{-|x|^2/2}$ (and then evaluated at $\pm\infty$) upon integrating by parts. Thus, $\mathbb{E}(\langle DF(\xi),u(\xi)\rangle) = \mathbb{E}(F(\xi)(\delta u)(\xi))$. □

Next, we prove an essential identity relating the two operators δ and D.

Proposition 11. *For any $F(\xi) \in \mathscr{H}_k$, $\delta DF(\xi) = kF(\xi)$*

Proof. As an element in the kth order Wiener chaos, write $F(\xi) = \prod_{i=1}^{N} h_{\alpha_i}(\xi_i)$, for some multi-index α. By definition of δ in (2.8)

$$\begin{aligned}
\delta DF(\xi) &= \langle DF(\xi),\xi\rangle - \text{div}\, DF(\xi) \\
&= \left\langle (\frac{\partial F(\xi)}{\partial x_1},\cdots,\frac{\partial F(\xi)}{\partial x_N}),(\xi_1,\cdots,\xi_N)\right\rangle - \text{div}\,(\frac{\partial F(\xi)}{\partial x_1},\cdots,\frac{\partial F(\xi)}{\partial x_N}) \\
&= (\xi_1\frac{\partial F(\xi)}{\partial x_1}+\cdots+\xi_N\frac{\partial F(\xi)}{\partial x_N}) - (\frac{\partial^2 F(\xi)}{\partial x_1^2}+\cdots+\frac{\partial^2 F(\xi)}{\partial x_N^2}) \\
&= \sum_{j=1}^{N}(\xi_j\frac{\partial F}{\partial x_j}-\frac{\partial^2 F}{\partial x_j^2})(\xi)
\end{aligned}$$

Note that $F(\xi) = \prod_{j=1}^{N} h_{\alpha_j}(\xi_j) = \prod_{i \neq j}^{N} h_{\alpha_i}(\xi) h_{\alpha_j}(\xi_j)$. We take its first and second-partial derivatives which are,

$$
\begin{aligned}
\frac{\partial F(\xi)}{\partial x_j} &= \prod_{i \neq j}^{N} \frac{\partial}{\partial x_j} \left(h_{\alpha_i}(\xi_i) h_{\alpha_j}(\xi_j) \right) = \prod_{i \neq j}^{N} h_{\alpha_i}(\xi_i) \frac{\partial}{\partial x_j} \left(h_{\alpha_j}(\xi_j) \right) \\
\frac{\partial^2 F(\xi)}{\partial x_j^2} &= \prod_{i \neq j}^{N} h_{\alpha_i}(\xi_i) \frac{\partial^2}{\partial x_j^2} \left(h_{\alpha_j}(\xi_j) \right)
\end{aligned}
$$

Substituting these derivatives into the last equation, we continue with

$$
\begin{aligned}
\delta DF(\xi) &= \sum_{j=1}^{N} \left[\left(\xi_j \prod_{i \neq j}^{N} h_{\alpha_i}(\xi_i) \frac{\partial}{\partial x_j} \left(h_{\alpha_j}(\xi_j) \right) \right) - \left(\prod_{i \neq j}^{N} h_{\alpha_i}(\xi_i) \frac{\partial^2}{\partial x_j^2} \left(h_{\alpha_j}(\xi_j) \right) \right) \right] \\
&= \sum_{j=1}^{N} \prod_{i \neq j}^{N} h_{\alpha_i}(\xi_i) \left[\xi_j \frac{\partial}{\partial x_j} \left(h_{\alpha_j}(\xi_j) \right) - \frac{\partial^2}{\partial x_j^2} \left(h_{\alpha_j}(\xi_j) \right) \right] \\
&= \sum_{j=1}^{N} \prod_{i \neq j}^{N} h_{\alpha_i}(\xi_i) \left[\xi_j h'_{\alpha_j}(\xi_j) - h''_{\alpha_j}(\xi_j) \right]
\end{aligned}
$$

Now, by using the second derivative identity (2.5) for the Hermite polynomial, we have

$$
\begin{aligned}
\delta DF(\xi) &= \sum_{j=1}^{N} \prod_{i \neq j}^{N} h_{\alpha_i}(\xi_i) \left[\xi_j h'_{\alpha_j}(\xi_j) - \xi_j h'_{\alpha_j}(\xi_j) + \alpha_j h_{\alpha_j}(\xi_j) \right] \\
&= \sum_{j=1}^{N} \prod_{i \neq j}^{N} h_{\alpha_i}(\xi_i) \alpha_j h_{\alpha_j}(\xi_j) \\
&= \sum_{j=1}^{N} \alpha_j \prod_{i \neq j}^{N} h_{\alpha_i}(\xi_i) h_{\alpha_j}(\xi_j) \\
&= \sum_{j=1}^{N} \alpha_j F(\xi) \\
&= F(\xi) k \\
&= k F(\xi)
\end{aligned}
$$

This ends the proof. $\square$

To end this section, let us make some intuitive comments on the two results that we obtained in this section. The equation (2.7) tells us that the divergence operator δ defined by (2.8) seems to act like an adjoint of the gradient operator D (but not quite an adjoint because the result has been proven only for functions F, u that are C^1 and has polynomial growth). The identity (11) can be viewed intuitively as: the effect of multiplying (or composition) the divergence δ with the gradient operator D on any $F \in \mathscr{H}_k$ is simply multiplying F by the scalar quantity k, which happens to be the order of the Wiener chaos $\mathscr{H}_k$.

Chapter 3

Main Results

This section contains the main theorem of this paper, a Central Limit Theorem applied to functionals of Gaussian processes. The important implication in this theorem is that conditon (3) implies condition (1) - this paves the way for the convergence of $\{F_n\}_{n\geq 1}$ to the Normal law.

Theorem 12 (Central Limit Theorem)**.** *Fix $k \geq 2$. Let $\{F_n\}_{n\geq 1}$ be a sequence in $\mathscr{H}_k$. Assume $\lim_{n\to\infty} \mathbb{E}({F_n}^2) = \sigma^2$. Then the following conditions are equivalent:*

1. $F_n \longrightarrow N(0,\sigma^2)$ *as* $n \longrightarrow \infty$

2. $\mathbb{E}({F_n}^4) \longrightarrow 3\sigma^4$ *as* $n \longrightarrow \infty$

3. $\|DF_n\|^2 \longrightarrow k\sigma^2$ *in* L^2 *as* $n \longrightarrow \infty$

Note that the first two conditions are convergences in distribution. We will prove that $(3) \Longrightarrow (1)$ and that $(1) \Longrightarrow (2)$. The proof of $(2) \Longrightarrow (3)$ can be obtained from the references [5] and [6].

Proof of $(3) \Longrightarrow (1)$. Suppose that $\sigma^2 = 1$. We assume $\|DF_n\|^2 \longrightarrow k$ in L^2 as $n \longrightarrow \infty$. We need to prove $F_n \longrightarrow N = N(0,1)$ as $n \longrightarrow \infty$ in distribution, which means that

$$\lim_{n\to\infty} \mathbb{E}[\varphi(F_n)] = \mathbb{E}[\varphi(N)] \tag{3.1}$$

for any function φ that is twice-differentiable, with continuous and bounded derivatives. To start the proof, fix a test function φ such that $\|\varphi''\|_\infty := \sup_x |\varphi(x)| < \infty$. For $0 \leq t \leq 1$, define the mapping

$$\psi(t) = \mathbb{E}\left[\varphi(\sqrt{t}F_n + \sqrt{1-t}N)\right].$$

Note that

$$\psi(1) = \mathbb{E}(\varphi(F_n)),\ \psi(0) = \mathbb{E}(\phi(N)).$$

Hence, by the Fundamental Theorem of Calculus,

$$\mathbb{E}(\varphi(F_n)) - \mathbb{E}(\varphi(N)) = \psi(1) - \psi(0) = \int_0^1 \psi'(t)dt \tag{3.2}$$

Let us compute the right-hand side. Taking the derivative of ψ with respect to t,

$$\begin{aligned}
\psi'(t) &= \frac{d}{dt}\mathbb{E}\left[\varphi(\sqrt{t}F_n + \sqrt{1-t}N)\right] \\
&= \mathbb{E}\left[\frac{d}{dt}\mathbb{E}\left[\varphi(\sqrt{t}F_n + \sqrt{1-t}N)\right]\right] \\
&= \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)(\frac{1}{2\sqrt{t}}F_n - \frac{1}{2\sqrt{1-t}}N)\right]
\end{aligned}$$

and so

$$\begin{aligned}
\int_0^1 \psi'(t)dt &= \int_0^1 \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)(\frac{1}{2\sqrt{t}}F_n - \frac{1}{2\sqrt{1-t}}N)\right] dt \\
&= \int_0^1 \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)\frac{1}{2}(\frac{F_n}{\sqrt{t}} - \frac{N}{\sqrt{1-t}})\right] dt \\
&= \frac{1}{2}\int_0^1 \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)F_n\right]\frac{dt}{\sqrt{t}} \\
&\quad -\frac{1}{2}\int_0^1 \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)N\right]\frac{dt}{\sqrt{1-t}} \\
&= \frac{1}{2}(A_n - B_n).
\end{aligned}$$

where

$$A_n = \int_0^1 \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)F_n\right]\frac{dt}{\sqrt{t}} \tag{3.3}$$

and

$$B_n = \int_0^1 \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)N\right]\frac{dt}{\sqrt{1-t}} \tag{3.4}$$

Now, we will work on each of the terms A_n and B_n. Starting with A_n, we notice that in the integrand of A_n, the argument of $\mathbb{E}$ contains an F_n term. By the identity in Proposition 11, we can write $F_n(\xi) = \frac{1}{k}\delta(DF_n(\xi))$ so that

$$\begin{aligned} A_n &= \frac{1}{k}\int_0^1 \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)\delta DF_n\right]\frac{dt}{\sqrt{t}} \\ &= \frac{1}{k}\int_0^1 \mathbb{E}\left[\left\langle D(\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)), DF_n\right\rangle\right]\frac{dt}{\sqrt{t}} \end{aligned}$$

where we have used Proposition 2.9. Next, the derivative D is with respect to ξ and so,

$$\begin{aligned} A_n &= \frac{1}{k}\int_0^1 \mathbb{E}\left[\left\langle D(\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)), DF_n\right\rangle\right]\frac{dt}{\sqrt{t}} \\ &= \frac{1}{k}\int_0^1 \mathbb{E}\left[\left\langle \varphi''(\sqrt{t}F_n + \sqrt{1-t}N)(\sqrt{t}DF_n), DF_n\right\rangle\right]\frac{dt}{\sqrt{t}} \\ &= \frac{1}{k}\int_0^1 \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}N)\|DF_n\|^2\right]dt \end{aligned}$$

where we applied the Chain Rule that is why the term $\sqrt{t}$ was cancelled. We stop here for A_n; this is the form that we want.

$$A_n = \frac{1}{k}\int_0^1 \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}N)\|DF_n\|^2\right]dt \tag{3.5}$$

Next, we work on B_n. Recall (3.4),

$$B_n = \int_0^1 \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)N\right]\frac{dt}{\sqrt{1-t}}.$$

This time, we cannot apply the same technique that we did for A_n because this time there is an N-term inside the argument of $\mathbb{E}$ in the integrand of B_n. However, N is the standard Normal law, and hence,

$$\begin{aligned} B_n &= \int_0^1 \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}N)N\right]\frac{dt}{\sqrt{1-t}} \\ &= \int_{\mathbb{R}} \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}x)\right] x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{1-t}}. \end{aligned}$$

Now we will apply the integration-by-parts technique. To this end, choose,

$$u = \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}x)\right] \text{ and } dv = xe^{-\frac{x^2}{2}}$$

so that

$$du = \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}x)\right]\sqrt{1-t}dx \text{ and } v = -e^{-\frac{x^2}{2}}.$$

Hence,

$$\begin{aligned} B_n &= \int_{\mathbb{R}} \mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}x)\right] x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{1-t}} \\ &= -\mathbb{E}\left[\varphi'(\sqrt{t}F_n + \sqrt{1-t}x)\right]e^{-\frac{x^2}{2}}\,\big|_{-\infty}^{+\infty} + \int_{\mathbb{R}} \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}x)\right]\sqrt{1-t}e^{-\frac{x^2}{2}}dx \\ &= \int_0^1 \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}N)\right]dt. \end{aligned}$$

This is the form that we want for B_n, that is,

$$B_n = \int_0^1 \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}N)\right]dt. \tag{3.6}$$

We can now combine equations (3.5) and (3.6) to compute

$$\begin{aligned}
\int_0^1 \psi'(t)\,dt &= \frac{1}{2}(A_n - B_n) \\
&= \frac{1}{2}\frac{1}{k}\int_0^1 \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}N)\cdot \|DF_n\|^2\right] dt \\
&\quad -\frac{1}{2}\int_0^1 \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}N)\right] dt \\
&= \frac{1}{2}\frac{1}{k}\|DF_n\|^2 \int_0^1 \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}N)\right] dt \\
&\quad -\frac{1}{2}\int_0^1 \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}N)\right] dt \\
&= \int_0^1 \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}N)\right] dt \cdot (\frac{1}{2k}\|DF_n\|^2 - \frac{1}{2}) \\
&= \frac{1}{2}\int_0^1 \mathbb{E}\left[\varphi''(\sqrt{t}F_n + \sqrt{1-t}N)\right] \cdot (\frac{1}{k}\|DF_n\|^2 - 1)dt
\end{aligned}$$

Using (3.2), we now have

$$\begin{aligned}
|\mathbb{E}(\varphi(F_n)) - \mathbb{E}(\varphi(N))| &= \left|\int_0^1 \psi'(t)dt\right| \\
&= \left|\frac{1}{2}\int_0^1 \mathbb{E}\left[\varphi''\left(\sqrt{t}F_n + \sqrt{1-t}N\right)\right]\cdot\left(\frac{1}{k}\|DF_n\|^2 - 1\right)\right| dt \\
&\leq \frac{1}{2}\int_0^1 \left|\mathbb{E}\left[\varphi''\left(\sqrt{t}F_n + \sqrt{1-t}N\right)\cdot\left(\frac{1}{k}\|DF_n\|^2 - 1\right)\right]\right| dt \\
&\leq \frac{1}{2}\|\varphi''\|_\infty \mathbb{E}\left(\left|\frac{1}{k}\cdot\|DF_n\|^2 - 1\right|\right).
\end{aligned}$$

Now, by assumption, it follows that the right-hand side of the above inequality goes to 0 as $n \to \infty$, because $\|\varphi''\|_\infty < \infty$ is independent of n. Thus, $|\mathbb{E}(\varphi(F_n)) - \mathbb{E}(\varphi(N))| \to 0$ also as $n \to \infty$, for any function φ. This is what we want (see 3.1). We have proven that the sequence $\{F_n\}_{n\geq 1}$ converges in law (or converges in distribution) to $N(0,1)$. Note that we have also provided a rate of convergence in a suitable metric,

$$\sup_{\{\varphi:\|\varphi''\|_\infty<1\}} |\mathbb{E}(\varphi(F_n))| \le \frac{1}{2}\mathbb{E}(|\frac{1}{k}\|DF_n\|^2 - 1|).$$

This is the end of the proof of (3) implies (1). It is important to note that we did not use the assumption

$$\lim_{n\to\infty} \mathbb{E}({F_n}^2) = \sigma^2$$

in this part of the proof.

Proof of $(1) \Longrightarrow (2)$. We assume that $F_n \longrightarrow N := N(0,\sigma^2)$ as $n \longrightarrow \infty$. We want to prove that $\mathbb{E}({F_n}^4) \longrightarrow 3\sigma^4$ as $n \longrightarrow \infty$. For $p \ge 2$, we note that we have

$$\mathbb{E}(|F_n|^p) \le c_k \mathbb{E}(|F_n|^2)$$

where $c_k \in \mathbb{R}$ that is independent of n.

Now, by hypothesis,

$$\lim_{n\to\infty} \mathbb{E}({F_n}^2) = \sigma^2$$

and so

$$\mathbb{E}(|F_n|^p) \le c_k \mathbb{E}(|F_n|^2) \longrightarrow c_k\sigma^2, \text{ as } n \to \infty$$

which implies that $\sup_n \mathbb{E}(|F_n|^p) < \infty$ for all $p \ge 2$. Now, by assumption, $F_n \longrightarrow N$ as $n \longrightarrow \infty$, so that by continuity of $\mathbb{E}$,

$$\mathbb{E}(F_n^4) \longrightarrow \mathbb{E}(N^4) \text{ as } n \to \infty.$$

Hence, to finish the proof, all we need to do now is show that

$$\mathbb{E}(N^4) = 3\sigma^4.$$

Let $X \sim N(0,\sigma^2)$ and $Y = \dfrac{X}{\sigma} \sim N(0,1)$. Then for any $n \geq 0$,

$$\mathbb{E}(N^n) = \mathbb{E}(X^n) = \sigma^n \mathbb{E}((\frac{X}{\sigma})^n) = \sigma^n \mathbb{E}(Y^n). \tag{3.7}$$

The characteristic function of Y is

$$\begin{aligned}
\phi(t) = \mathbb{E}(e^{itY}) &= e^{-\frac{t^2}{2}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} t^n \\
&= 1 - \frac{t^2}{2\cdot 1!} + \frac{t^4}{2^2 \cdot 2!} - \frac{t^6}{2^3 \cdot 3!} + \ldots
\end{aligned}$$

where

$$\phi^n(0) = \begin{cases} 0, & n = 2k+1 \\ \dfrac{(-1)^k (2k)!}{2^k k!}, & n = 2k \end{cases}$$

Now,

$$\mathbb{E}(Y^n) = \frac{\phi^{(n)}(0)}{i^n} = \begin{cases} 0, & n = 2k+1 \\ \dfrac{(2k)!}{2^k k!} = n!! = 1\cdot 3\cdot 5\cdot 7\cdot (n-1), & n = 2k \end{cases}$$

In particular, for $n = 4$,

$$\mathbb{E}(Y^4) = \frac{4!}{2^2 2!} = 3.$$

Thus, from (3.7), we have

$$\mathbb{E}(N^4) = \sigma^4 \mathbb{E}(Y^4) = 3\sigma^4.$$

This ends the proof of the Central Limit Theorem for functionals $\{F_n\}_{n\geq 1}$ in the kth Wiener chaos.

Chapter 4

The Infinite Dimensional Case

In this section, we will extend the notations and results of the previous sections to the case of infinite dimension. The main discussion in the first-subsection is the extension of the Wiener chaos in infinite dimension. The second sub-section shows how to extend the operators D and δ by using a truncation process and then by passing to a limit. Also, this sub-section discusses how to extend the identity given in Proposition 11 to its infinite-dimensional case. Finally, the third sub-section proves an important equation that is related to the third condition in the Central Limit Theorem.

4.1 Preliminary Extensions

Let us recall the notations and results in the finite-dimensional case. Let N be a fixed positive integer.

1. $\xi = (\xi_1, \cdots, \xi_N)$ denotes the N-dimensional random vector where the ξ_i's are independent and $\xi_i \sim N(0,1)$.

2. $\mathsf{X}_\alpha = \prod_{i=1}^{N} h_{\alpha_i}(\xi_i)$, where $\alpha = (\alpha_1, \cdots, \alpha_N), \alpha_i \in \{0, 1, 2, ...\}$, is the family of random variables generated from the sequence of Hermite polynomials $h_k(x)$. Hermite polynomials are defined in (2.1) in Section 2.1.

3. $\mathscr{H}_k$ which is the linear span of $\{\mathsf{X}_\alpha : |\alpha| = \alpha_1 + \cdots + \alpha_N = k,\ \alpha_i \geq 0\}$, is the Wiener chaos of order k.

4. We have the orthogonal decomposition $\mathsf{L}^2(\Omega) = \bigoplus_{k=0}^{\infty} \mathscr{H}_k$ where $\mathscr{H}_0 = \mathbb{R}$.

5. Given real-valued function $F : \mathbb{R}^N \longrightarrow \mathbb{R}$ and vector-valued function $u : \mathbb{R}^N \to \mathbb{R}^N$, both defined on $\mathbb{R}^N$, with F, u being C^1 functions with polynomial growth, we have the following results:

   (a) $\mathbb{E}(\langle DF(\xi), u(\xi)\rangle) = \mathbb{E}(F(\xi)\delta(u(\xi)))$

   (b) $\delta(DF(\xi)) = kF(\xi)$.

   where $DF = (\frac{\partial F}{x_1}, ..., \frac{\partial F}{x_N})$ and $(\delta u)(\xi) = \langle u(\xi), \xi\rangle - \operatorname{div} u(\xi)$.

6. The Central Limit Theorem. Fix $k \geq 2$. Let $\{F_n\}_{n\geq 1}$ be a sequence in $\mathscr{H}_k$. If $\mathbb{E}(F_n^2) \longrightarrow \sigma^2$ then the following conditions are equivalent:

   (a) $F_n \longrightarrow N(0, \sigma^2)$ as $n \longrightarrow \infty$

   (b) $\mathbb{E}({F_n}^4) \longrightarrow 3\sigma^4$ as $n \longrightarrow \infty$

   (c) $\|DF_n\|^2 \longrightarrow k\sigma^2$ in L^2 as $n \longrightarrow \infty$.

Now, consider an infinite-dimensional random vector $\xi = (\xi_1, \cdots, \xi_N, \cdots)$ where ξ_i's are independent and $\xi_i \sim N(0,1)$. As in the finite-dimensional case, we define a family $\{\mathsf{X}_\alpha\}_\alpha$ of random variables generated from the sequence of Hermite polynomials by

$$\mathsf{X}_\alpha = \prod_{i=1}^{\infty} h_{\alpha_i}(\xi_i),$$

where α is an infinite multi-index $\alpha = (\alpha_1, \ldots, \alpha_N, \ldots), \alpha_i \in \{0, 1, 2, ...\}$, and most importantly,

$$\alpha_n = 0, n \geq n_0 \text{ for some } n_0 \in \mathbb{N},$$

that is, only finitely many $\alpha_i' s$ are non-zeros. Then X_α is well-defined because the product contains only a finite number of terms, taking into account that $h_{\alpha_n}(x) = h_0(x) = 1$ for $n \geq n_0$. The notations for the multi-index α remain the same, like,

$$|\alpha| = \sum_{i=1}^{\infty} \alpha_i \text{ and } \alpha! = \prod_{i=1}^{\infty} (\alpha_i!).$$

As in the finite-dimensional case, these X_α's are used to define a Wiener chaos of order k, as follows:

Definition 13. *For each integer $k \geq 1$, the kth-order Wiener chaos is*

$$\mathscr{H}_k = \overline{span}\{\mathsf{X}_\alpha : |\alpha| = k\},$$

where $\bar{X}$ means the closure of X.

For example, for $k = 1$,

$$\mathscr{H}_1 = \overline{\text{span}}\{\xi_1, \xi_2, \dots, \xi_N, \dots\} \tag{4.1}$$

where α can be any of the following

$$(1, 0, 0, \dots, 0, \dots), (0, 1, 0, \dots, 0, 0, \dots), ...$$

This example, $\mathscr{H}_1$, will be used in Section 5, where we discuss some examples. For $k = 2$,

$$\mathscr{H}_2 = \overline{\text{span}}\{\mathsf{X}_\alpha : |\alpha| = 2\}.$$

where α can be any of the following

$$(1,1,0,0,\ldots,0,\ldots),(1,0,1,0,\ldots,0,\ldots),(0,1,1,0,\ldots,0,\ldots),$$

$$(2,0,0,0,\ldots,0,\ldots),(0,2,0,0,\ldots,0,\ldots)...$$

Hence, the X_α's in $\mathscr{H}_2$ are

$$\begin{aligned}
\mathsf{X}_{(1,1,0,0,\ldots)} &= \prod_{i=1}^{\infty} h_{\alpha_i}(\xi_i) = \xi_1 \cdot \xi_2 \\
\mathsf{X}_{(1,0,1,0,\ldots)} &= \prod_{i=1}^{\infty} h_{\alpha_i}(\xi_i) = \xi_1 \cdot \xi_3 \\
&\vdots \\
\mathsf{X}_{(2,0,0,0,\ldots)} &= \frac{1}{2}({\xi_1}^2 - 1) \\
\mathsf{X}_{(0,2,0,0,\ldots)} &= \frac{1}{2}({\xi_2}^2 - 1)\ldots
\end{aligned}$$

Hence, we have

$$\mathscr{H}_2 = \overline{\text{span}}\{\xi_i\xi_j, {\xi_i}^2 - 1, \text{ where } i \neq j\ , i,j = 1,2,\ldots\}.$$

Note that an infinite basis for $\mathscr{H}_k$ is $\{\mathsf{X}_\alpha : |\alpha| = k\}$. With these extended $\mathscr{H}_k$'s, the orthogonal decomposition of $\mathsf{L}^2(\Omega)$ still holds, that is

$$\mathsf{L}^2(\Omega) = \bigoplus_{k=0}^{\infty} \mathscr{H}_k, \mathscr{H}_0 = \mathbb{R}$$

and in fact, the proof in [5] holds in this more general case.

4.2 The operators D and δ and an identity

The next job is to have analogies for D and δ in the infinite-dimensional setting and then to derive an identity analogous to the finite-dimensional case, as in Proposition 11. Consider the space

$$\mathscr{S} = \{F(\xi) : F \in C^1(\mathbb{R}^N \to \mathbb{R}), \text{ for some } N \geq 1, F \text{ and } \frac{\partial F}{\partial x_i}\text{'s have polynomial growth}\}.$$

It is clear that $\mathscr{S}$ is a linear space. For any $F \in \mathscr{S}$, define D

$$DF(\xi) = \left(\frac{\partial F}{\partial x_1}(\xi), \ldots, \frac{\partial F}{\partial x_N}(\xi), 0, 0, \ldots\right)$$

The space $\mathscr{S}$ is a normed linear space with respect to the norm $\|\cdot\|_{1,2}$ defined by

$$\|F\|_{1,2} = \sqrt{\mathbb{E}(F^2) + \mathbb{E}(\|DF\|_2^2)}$$

where

$$\|DF\|_2^2 = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial x_i}(\xi)\right)^2.$$

Note that each $\mathbb{E}\left(\frac{\partial F}{\partial x_i}(\xi)\right)^2$ is finite because F is C^1 and each $\frac{\partial F}{\partial x_i}$ is continuous and has polynomial growth.

The space $\mathscr{S}$ may not be complete and so we consider its norm-closure. Let $\mathscr{D}^{1,2} = \overline{\mathscr{S}}^{\|\cdot\|_{1,2}}$ be the closure of $\mathscr{S}$ with respect to $\|\cdot\|_{1,2}$. Thus, we consider the operator D as a mapping from $\mathscr{D}^{1,2} \to L^2(\Omega, l^2)$, where $l^2 = l^2(\mathbb{R})$ is the Hilbert space of real-square-summable sequences,

$$\{\{a_n\}_{n\geq 1} : a_n \in \mathbb{R}, \|a_n\|_2^2 < \infty\}, \text{ where } \|a_n\|_2 = \sqrt{\sum_{n=1}^{\infty} |a_n|^2}$$

By definition, for any $X \in \mathscr{D}^{1,2}$, there exists a sequence $X_n \in \mathscr{S}$ such that

$$X_n \longrightarrow X \text{ as } n \to \infty \text{ in } L^2(\Omega)$$

and DX_n also converges in $L^2(\Omega; l^2)$.

Next, we develop the infinite-dimensional analogy for the operator δ, see (2.8), that was defined in Section 2. The linear operator D was defined above. Define δ to be the (linear) adjoint of D. The domain of δ is

$$Dom\ \delta = \{u \in L^2(\Omega; l^2) : \text{for any } F \in \mathscr{D}^{1,2}, |\mathbb{E}(\langle u, DF \rangle)| \leq c_u \|F\|_2\}.$$

To show the existence of δ, let $u \in Dom\ \delta$. Then, by definition, for any $F \in \mathscr{D}^{1,2}$, the linear functional

$$F \to \mathbb{E}(\langle DF, u \rangle)$$

is continuous in $L^2(\Omega)$. Thus, there exists $\delta(u) \in L^2(\Omega)$ such that $\mathbb{E}(\langle DF, u \rangle_2) = \mathbb{E}(F \cdot \delta(u))$.

Now that we have our operators D and δ, we want to prove the identity given in Proposition 11 in the infinite-dimensional case. To prove this, we approximate X by a sequence $X_n \in \mathscr{S}$ in such a way that,

$$DX_n \longrightarrow DX \text{ as } n \longrightarrow \infty.$$

and

$$\delta(DX_n) \longrightarrow \delta(DX) \text{ as } n \longrightarrow \infty.$$

Then, the identity

$$\delta(DX) = kX,$$

will follow from $\delta(DX_n) = kX_n$. This ends the discussion of the extensions of Subsection 2.3, that is, the two operators and an identity that relates them. Since we have now laid down the infinite-dimensional groundwork for the proof of our Central Limit Theorem, the arguments in Section 3 can be lifted into the infinite-dimensional setting.

4.3 Behavior of the Derivative in $\mathscr{H}_k$

In this sub-section, we look at some properties of the derivative operator D. The first proposition shows that the family $\{D\mathsf{X}_\alpha\}_{\alpha \geq 0}$ is orthogonal

Proposition 14. *For multi-indices $\alpha \neq \alpha'$,*

$$\mathbb{E}(\langle D\mathsf{X}_\alpha, D\mathsf{X}_{\alpha'}\rangle_2) = 0$$

Proof.

$$\begin{aligned}
\mathbb{E}(\langle D\mathsf{X}_\alpha, D\mathsf{X}_{\alpha'}\rangle_2) &= \mathbb{E}\left(\sum_{i=1}^{\infty} (D\mathsf{X}_\alpha)_i (D\mathsf{X}_{\alpha'})_i\right) \\
&= \sum_{i=1}^{\infty} \mathbb{E}\left(\frac{\partial \mathsf{X}_\alpha(\xi)}{\partial x_i} \cdot \frac{\partial \mathsf{X}_{\alpha'}(\xi)}{\partial x_i}\right) \\
&= \sum_{i=1}^{\infty} \mathbb{E}\left(\prod_{i=1}^{\infty} \frac{\partial h_{\alpha_i}(\xi_i)}{\partial x_i} \cdot \frac{\partial h_{\alpha_i'}(\xi_i)}{\partial x_i}\right) \\
&= \sum_{i=1}^{\infty} \mathbb{E}\left(\prod_{i=1}^{\infty} h'_{\alpha_i}(\xi_i) h'_{\alpha_i'}(\xi_i)\right) \\
&= \sum_{i=1}^{\infty} \mathbb{E}\left(\prod_{i=1}^{\infty} h_{\alpha_i - 1}(\xi_i) h_{\alpha_i' - 1}(\xi_i)\right) \\
&= 0
\end{aligned}$$

by orthogonality of the Hermite polynomials since $\alpha_i - 1 \neq \alpha_i' - 1$ for at least one index i. Also, we have used the Appell-sequence property of the Hermite polynomial, see (2.3), in order to change the derivatives of h_α's into $h_{\alpha-1}$. $\square$

Theorem 15. *Suppose $X \in \mathscr{H}_k$ for some fixed k. Then we have the following results:*

1. $X \in \mathscr{D}^{1,2}$
2. $DX = \sum_{|\alpha|=k} c_\alpha D\mathsf{X}_\alpha$ *for some* c_α
3. $\mathbb{E}(||DX||_2^2) = k\mathbb{E}(||X||_2^2)$

Proof. Suppose $X \in \mathscr{H}_k$. We can express X as a linear combination of the elements in the set $\{\mathsf{X}_\alpha : |\alpha| = k, \alpha_i \geq 0\}$:

$$X = \sum_{|\alpha|=k} c_\alpha \mathsf{X}_\alpha \text{ for some } c_\alpha,$$

where

$$\mathsf{X}_\alpha = \prod_{i=1}^{\infty} h_{\alpha_i}(\xi_i).$$

We define a truncated X via a mapping r, which is defined in the following way. For each multi-index α, define a mapping $r : \mathbb{N}^\infty \to \mathbb{N}$ by

$$r(\alpha) := \max_i \{\alpha_i \neq 0\}.$$

For example, if $\alpha = (1,2,0,4,5,0,7,8,0,0,1,2,0,0,...)$ then $r(\alpha) = 12$.

Definition 16. *Fix N. For $X \in \mathscr{H}_k$, define the truncated X by*

$$X^N = \sum_{\alpha : r(\alpha) \leq N} c_\alpha \mathsf{X}_\alpha.$$

Then $X^N \in \mathscr{S}$ and we know that $X^N \to X$ in $L^2(\Omega)$. We will show that DX^N converges in $L^2(\Omega, l^2)$. This implies that X belongs to $\mathscr{D}^{1,2}$ and DX is the limit of DX^N. This will follow from the next proposition:

Proposition 17.

$$\mathbb{E}(\left\|DX^N\right\|_2^2) = k\mathbb{E}(\|X^N\|^2)$$

Proof. For some c_α, we can write

$$DX^N = \sum_{\alpha : r(\alpha) \leq N} c_\alpha D\mathsf{X}_\alpha. \tag{4.2}$$

First, we compute $\mathbb{E}(\|D\mathsf{X}_\alpha\|_2^2)$, that is, we compute the expectation of the square of the 2-norm for each individual term X_α. Noting that X^N is a truncation of X via the mapping $r(\alpha)$, we will then compute $\mathbb{E}(\left\|DX^N\right\|_2^2)$. For each α, we have

$$\begin{aligned}
\mathbb{E}(\|D\mathsf{X}_\alpha\|_2^2) &= \mathbb{E}\left(\sum_{i=1}^{\infty}\left(\frac{\partial \mathsf{X}_\alpha}{\partial x_i}(\xi)\right)^2\right) \\
&= \sum_{i=1}^{\infty}\mathbb{E}\left(\left(\frac{\partial \mathsf{X}_\alpha}{\partial x_i}(\xi)\right)^2\right) \\
&= \sum_{i=1}^{\infty}\mathbb{E}\left(\left(\frac{\partial}{\partial x_i}\left(\prod_{i=1}^{\infty} h_{\alpha_i}(\xi_i)\right)\right)^2\right).
\end{aligned}$$

Then we apply the product rule expansion for the derivative. Using the orthogonality of the Hermite polynomials, this expansion will be reduced so that we only consider indices $i \neq j$:

$$\begin{aligned}\mathbb{E}(\|D\mathsf{X}_\alpha\|_2^2) &= \sum_{i=1}^{\infty}\mathbb{E}\left(\left(\prod_{i\neq j}^{\infty}h_{\alpha_j}(\xi_i)h'_{\alpha_i}(\xi_i)\right)^2\right)\\ &= \sum_{i=1}^{\infty}\mathbb{E}\left(\left(\prod_{i\neq j}^{\infty}h_{\alpha_j}(\xi_i)h_{\alpha_i-1}(\xi_i)\right)^2\right),\end{aligned}$$

where we have used the Appell-property (2.3) of the Hermite polynomials again. Then we take the squares and use a property of the operator $\mathbb{E}$:

$$\begin{aligned}\mathbb{E}(\|D\mathsf{X}_\alpha\|_2^2) &= \sum_{i=1}^{\infty}\mathbb{E}\left(\prod_{i\neq j}^{\infty}h_{\alpha_j}^2(\xi_j)h_{\alpha_i-1}^2(\xi_i)\right)\\ &= \sum_{i=1}^{\infty}\left(\prod_{i\neq j}^{\infty}\mathbb{E}\left(h_{\alpha_j}^2(\xi_j)\right)\mathbb{E}\left(h_{\alpha_i-1}^2(\xi_i)\right)\right)\\ &= \sum_{i=1}^{\infty}\left(\prod_{i\neq j}^{\infty}(\alpha_j)!^{-1}(\alpha_i-1)!^{-1}\right)\end{aligned}$$

where we use Proposition 5.

$$\begin{aligned}\mathbb{E}(\|D\mathsf{X}_\alpha\|_2^2) &= \sum_{i=1}^{\infty}\frac{1}{\prod_{i\neq j}^{\infty}(\alpha_j)!(\alpha_i-1)!}\\ &= \sum_{i=1}^{\infty}\frac{1}{\prod_{i\neq j}^{\infty}(\alpha_j)!(\alpha_i-1)!}\cdot\frac{\alpha_i}{\alpha_i}\\ &= \sum_{i=1}^{\infty}\frac{\alpha_i}{\prod_{j=1}^{\infty}\alpha_j!}\\ &= \frac{\sum_{i=1}^{\infty}\alpha_i}{\prod_{j=1}^{\infty}\alpha_j!}\\ &= \frac{k}{\alpha!},\end{aligned}$$

where k is the order of the multi-index (also of the Wiener chaos). Thus, for each multi-index α, we have

$$\mathbb{E}(\|D\mathsf{X}_\alpha\|_2^2) = \frac{k}{\alpha!} \tag{4.3}$$

Now that we know the result for each α, we then put them altogether, noting that the expansion for DX^N is (4.2). Thus,

$$\mathbb{E}(\left\|DX^N\right\|_2^2) = \mathbb{E}(\|\sum_{\alpha:r(\alpha)\le N} c_\alpha D\mathsf{X}_\alpha\|_2^2) = \mathbb{E}(\sum_{\alpha:r(\alpha)\le N} c_\alpha^2 \|D\mathsf{X}_\alpha\|_2^2)$$

where we have used a Pythagorean identity for the 2-norm because the family $\{D\mathsf{X}_\alpha\}_\alpha$ is an orthogonal family (see Proposition 14). Now, using (4.3) for each α, we have

$$\begin{aligned}
\mathbb{E}(\|DX^N\|_2^2) &= \sum_{\alpha:r(\alpha)\le N} c_\alpha^2 \mathbb{E}(\|D\mathsf{X}_\alpha\|_2^2) \\
&= \sum_{\alpha;r(\alpha)\leqslant N} c_\alpha^2 \frac{k}{\alpha!} \\
&= k \sum_{\alpha;r(\alpha)\leqslant N} \frac{c_\alpha^2}{\alpha!} \\
&= k\mathbb{E}\left((X^N)^2\right)
\end{aligned}$$

In the last line, we note that we use the fact that

$$\mathbb{E}\left(\|X^N\|^2\right) = \sum_{\alpha;r(\alpha)\leqslant N} \frac{c_\alpha^2}{\alpha!},$$

which follows from Proposition (5) using the Pythagorean identity argument, because the family $\{\mathsf{X}_\alpha\}_\alpha$ is orthogonal (see Proposition 4):

$$\begin{aligned}
\mathbb{E}(\|X^N\|_2^2) &= \mathbb{E}\left(\left\|\sum_{\alpha:r(\alpha)\leq N} c_\alpha \mathsf{X}_\alpha\right\|_2^2\right) \\
&= \mathbb{E}(\sum_{\alpha:r(\alpha)\leq N} c_\alpha^2 \|\mathsf{X}_\alpha\|_2^2) \\
&= \sum_{\alpha:r(\alpha)\leq N} c_\alpha^2 \mathbb{E}(\|\mathsf{X}_\alpha\|_2^2) \\
&= \sum_{\alpha;r(\alpha)\leqslant N} \frac{c_\alpha^2}{\alpha!}
\end{aligned}$$

□

Finally, since we now have

$$\mathbb{E}(\|DX^N\|_2^2) = k\mathbb{E}(\|X^N\|^2) < \infty,$$

we must then have

$$DX^N \to DX,$$

and also parts (2) and (3) of the Theorem hold. □

Chapter 5

Applications

In this section, we apply our Central Limit Theorem (CLT) in Section 3 to some examples. In particular, given a sequence $\{F_n\}_{n\geq 1}$, we will use the fact that the third condition implies the first condition to show that F_n converges in distribution to the normal law N. The first example proves the classical (CLT) using our Central Limit Theorem. The second example is an application using Gaussian Moving Averages.

5.1 The Classical Central Limit Theorem

Recall the classical CLT:

Theorem 18. *Let $\{X_n\}_{n\geq 1}$ be a sequence of independent, identically distributed random variables, such that $\mathbb{E}(X_1^2) < \infty$. Set $m = \mathbb{E}(X_1)$ and $\sigma = Var(X_1)$. Then,*

$$\frac{X_1 + X_2 + \ldots + X_n - nm}{\sigma\sqrt{n}} \to N(0,1) \textit{ in distribution.}$$

We will illustrate the classical CLT using our CLT in Section 3. For a theorem of such fundamental importance to probability, the classical CLT has a simple proof using characteristic functions. This is the classical proof:

Proof. For any random variable, Y, with zero mean and unit variance $(var(Y)=1)$, the characteristic function of Y is, by Taylor's theorem,

$$\varphi_Y(t) = 1 - \frac{t^2}{2} + o(t^2) \text{ as } t \to 0$$

where $o(t^2)$ is "little o-notation" for some function of t that goes to zero more rapidly than t^2. Letting Y_i be $(X_i - \mu)/\sigma$, the standardized value of X_i, it is easy to see that the standardized mean of the observations $X_1, X_2, \ldots, X_n$ is

$$Z_n = \frac{n\overline{X}_n - n\mu}{\sigma\sqrt{n}} = \sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}}.$$

By properties of characteristic functions, the characteristic function of Z_n is

$$\left[\varphi_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \to e^{-t^2/2}, \text{ as } n \to \infty.$$

But this limit is just the characteristic function of a standard normal distribution $N(0,1)$, and the central limit theorem follows from the Levy continuity theorem, which confirms that the convergence of characteristic functions implies convergence in distribution. $\square$

The classical Central Limit Theorem is also known as the Second Fundamental Theorem of Probability. The First Fundamental Theorem is the Law of Large Numbers (LLN):

Theorem 19. *Let $\{X_n\}_{n\geq 1}$ be a sequence of independent, identically distributed random variables, such that $\mathbb{E}(|X_1|) < \infty$. Then,*

$$\frac{X_1 + \ldots + X_n}{n} \to m \textit{ almost surely}$$

where $m = \mathbb{E}(X_1)$. *Recall that* $X_n \to X$ almost surely *if* $\lim_{n\to\infty} X_n(\omega) = X(\omega)$, *for any* $\omega \notin N$ *where* $P(N) = 0$.

In conjunction with the Law of Large Numbers, we will illustrate the classical CLT using our CLT in Section 3.

One part of our CLT states that: For a sequence $\{F_n\}_{n\geq 1}$ in $\mathscr{H}_k$ for some k and assuming that $\mathbb{E}({F_n}^2) \longrightarrow \sigma^2$ as $n \to \infty$, we have:

- if $\|DF_n\|_2^2 \longrightarrow k\sigma^2$ in L^2 as $n \to \infty$
- then $F_n \longrightarrow N(0, \sigma^2)$ as $n \to \infty$ in distribution.

The key in illustrating the classical CLT is to finding an appropriate sequence of functionals $F_n \in \mathscr{H}_k$ that will satisfy two things:

1. $\mathbb{E}({F_n}^2) \longrightarrow \sigma^2$ as $n \to \infty$
2. $\|DF_n\|_2^2 \longrightarrow k\sigma^2$ in L^2 as $n \to \infty$

Fix $k \geq 1$. Let $\xi = (\xi_n)_{n\geq 1}$ be a sequence of Gaussian random variables such that $\xi_i \sim N(0,1)$ and are independent. Define

$$F_n(\xi) = \frac{1}{\sqrt{n}}(h_k(\xi_1) + h_k(\xi_2) + \cdots + h_k(\xi_n))$$

where k is fixed and $k \geq 1$. Here are some observations about F_n in this example.

1. $\mathbb{E}(F_n^2) = \sigma^2$. Let us prove this.

$$\begin{aligned}
\mathbb{E}({F_n}^2) &= \mathbb{E}[(\frac{1}{\sqrt{n}}(h_k(\xi_1)+h_k(\xi_2)+\cdots+h_k(\xi_n)))^2] \\
&= \mathbb{E}\left[\frac{1}{n}(h_k(\xi_1)+h_k(\xi_2)+\cdots+h_k(\xi_n))^2\right] \\
&= \frac{1}{n}\cdot n\cdot\mathbb{E}(h_k(\xi_1^2)) \\
&= \frac{1}{k!} = \sigma^2,
\end{aligned}$$

where we have applied Theorem 3.

2. Note that $\frac{\partial}{\partial x_i}F_n = \frac{1}{\sqrt{n}}h_k'(\xi_i)$, $1 \leqslant i \leqslant n$. Since $DF_n = (\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n})$, we define $D^iF_n = (0,\ldots,\frac{\partial F}{\partial x_i},0,\ldots,0)$. Now we can compute the square of the L^2-norm of DF_n:

$$\begin{aligned}
\|DF_n\|_2^2 &= \left[(D^1F_n)^2+(D^2F_n)^2+\cdots+(D^nF_n)^2\right] \\
&= \frac{1}{n}\left[(h_k'(\xi_1))^2+(h_k'(\xi_2))^2+\cdots+(h_k'(\xi_n))^2\right] \\
&= \frac{1}{n}\left[h_{k-1}^2(\xi_1)+h_{k-1}^2(\xi_2)+\cdots+h_{k-1}^2(\xi_n)\right]
\end{aligned}$$

By LLN, this converges almost surely in L^2 to $\mathbb{E}(h_{k-1}^2(\xi_1))$, that is,

$$\|DF_n\|_2^2 \longrightarrow \mathbb{E}(h_{k-1}^2(\xi_1)) \text{ as } n\to\infty.$$

Then, computing the limit $\mathbb{E}(h_{k-1}^2(\xi_1))$, we have

$$\begin{aligned}\mathbb{E}(h_{k-1}^2(\xi_1)) &= \frac{1}{(k-1)!}\mathbb{E}(\xi^2)\\ &= \frac{1}{(k-1)!}\\ &= k\sigma^2,\end{aligned}$$

where we have used the fact that $\mathbb{E}(\xi^2) = 1)$.

Thus, by the Central Limit Theorem (12), we have

$$F_n \longrightarrow N(0,1) \text{ as } n \longrightarrow \infty \text{ in distribution.}$$

Let us illustrate the classical Central limit Theorem by considering the case $k = 1$. Then from example (4.1), we have

$$\mathscr{H}_1 = \bar{\text{span}}\{\xi_1, \xi_2, \cdots, \xi_n, \ldots\}.$$

We define F_n by

$$F_n(\xi) = \frac{1}{\sqrt{n}}\left[h_1(\xi_1) + \cdots + h_1(\xi_n)\right] = \frac{1}{\sqrt{n}}\left[\xi_1 + \xi_2 + \cdots + \xi_n\right].$$

Since $\xi_i \sim N(0,1), \forall i$, then $\sum_{i=1}^n \xi_i \sim N(0,1)$. Thus $\mathbb{E}(F_n) = 0$ and $\mathbb{E}({F_n}^2) = 1$.

5.2 Gaussian Moving Averages

For an introduction to Gaussian Moving Averages, the reader may refer to [7] and [8].

5.2.1 An Auto-regressive Model

Consider a sequence $\{Z_n\}_{n\geq 0}$ of Gaussian, independent random variables with $\mathbb{E}(Z_n) = 0$ for $n \geq 0$ and

$$Var(Z_n) = \begin{cases} \dfrac{\sigma^2}{1-\lambda^2}, & n = 0 \\ \sigma^2, & n \geq 1 \end{cases} \tag{5.1}$$

where $\lambda^2 < 1$. For example, with $\xi = (\xi_0, \xi_1, ..., \xi_n, ...)$, we can take

$$Z_n = \begin{cases} \dfrac{1}{\sqrt{1-\lambda^2}}\xi_0, & n = 0 \\ \xi_n, & n \geq 1. \end{cases}$$

Define the iterative process

$$\begin{cases} X_0 &= Z_0, \\ X_n &= \lambda X_{n-1} + Z_n, \ \ n \geq 1 \end{cases}$$

The process $\{X_n\}_{n\geq 0}$ is called a first-order autoregressive process. It says that the state X_n at time n is a constant multiple of the state at time $n-1$ plus a random error term Z_n, whose expectation and variance are as described above. Doing the iteration yields

$$\begin{aligned} X_n &= \lambda(\lambda X_{n-2} + Z_{n-1}) + Z_n \\ &= \lambda^2 X_{n-2} + \lambda Z_{n-1} + Z_n \\ &\vdots \\ &= \textstyle\sum_{j=0}^{n} \lambda^{n-j} Z_j \\ &= \textstyle\sum_{j=0}^{n} c_{n-j} Z_j, \end{aligned}$$

where $c_{n-j} = \lambda^{n-j}$ or $c_j = \lambda^j$. In particular, this relationship between X_n and Z_n shows that $\mathbb{E}(X_n) = 0$ for every $n \geq 0$ because $\mathbb{E}(Z_n) = 0$. In other words, the iteration X_n has not changed the mean of Z_n. More generally, we have:

Lemma 20. *The process $\{X_n\}_{n\geq 1}$ is stationary.*

Proof. A stationary process is a stochastic process whose joint probability distribution does not change when shifted in time or space. To show that it is stationary, we will show that its $Cov\,(X_n, X_{n+m})$ depends only on m:

$$\begin{aligned}
Cov(X_n, X_{n+m}) &= Cov(\sum_{i=0}^{n} \lambda^{n-i} Z_i, \sum_{i=0}^{n+m} \lambda^{n+m-i} Z_i) \\
&= \sum_{i=0}^{n} \lambda^{n-1} \lambda^{n+m-i} Cov(Z_i, Z_i) \\
&= \sum_{i=0}^{n} \lambda^{n-1} \lambda^{n+m-i} Var(Z_i) \\
&= \lambda^{2n+m} Var(Z_0) + \sum_{i=1}^{n} \lambda^{n-1} \lambda^{n+m-i} Var(Z_i)
\end{aligned}$$

Now, by our choice of $\{Z_n\}_{n\geq 1}$, we use its variance (5.1) so that

$$\begin{aligned}
Cov(X_n, X_{n+m}) &= \frac{\lambda^{2n+m}\sigma^2}{1-\lambda^2} + \sum_{i=1}^{n} \lambda^{2n+m-2i} \sigma^2 \\
&= \sigma^2 \lambda^{2n+m} \left[\frac{1}{1-\lambda^2} + \sum_{i=1}^{n} \lambda^{-2i} \right] \\
&= \sigma^2 \lambda^{2n+m} \left[\frac{1}{1-\lambda^2} + \lambda^{-2} (\frac{1-\lambda^{-2n}}{1-\lambda^{-2}}) \right] \\
&= \sigma^2 \lambda^{2n+m} \left[\frac{1}{1-\lambda^2} + \frac{\lambda^{2n}-1}{\lambda^{2n}(\lambda^2-1)} \right] \\
&= \sigma^2 \lambda^{2n+m} \left[\frac{-1}{\lambda^{2n}(\lambda^2-1)} \right] \\
&= \frac{\sigma^2 \lambda^m}{1-\lambda^2}
\end{aligned}$$

where the preceding uses the fact that Z_i and Z_j are uncorrelated when $i \neq j$. As $\mathbb{E}(X_n) = 0$ for every n and that the covariance only depends on n, hence we see that $\{X_n\}_{n\geq 0}$ is stationary. $\square$

The above proof also shows that the variance of X_n is

$$\mathbb{E}(X_n^2) = Cov(X_n, X_n) = \frac{\sigma^2}{1-\lambda^2}.$$

For every m, define

$$K(m) = \frac{\lambda^m}{1-\lambda^2}.$$

Using this notation, we have

$$Cov(X_n, X_{n+m}) = K(m), \text{ when } \sigma^2 = 1.$$

In terms of $c_j's$, we can expand $K(m)$ as,

$$\begin{aligned} K(m) &= Cov(X_n, X_p) \\ &= \sum_{j=1}^{n} c_{n-j}c_{p-j} + \left(\frac{c_n c_p}{1-\lambda^2}\right), \end{aligned}$$

where $p = n+m$. We will see that the process $\{X_n\}_{n\geq 0}$ is ergodic. A central aspect of ergodic theory is about the behavior of a dynamical system when it is allowed to run for a long time. This is expressed through ergodic theorems which assert that, under certain conditions, the time average of a function along the trajectories exists almost everywhere and is related to the space average.

Accordingly, if $\mathbb{E}(f(x_n)) < \infty$ and

$$\frac{f(x_1) + \cdots + f(x_n)}{n} \longrightarrow Y = \mathbb{E}(f(x_1)|\mathfrak{F}_\infty)$$

where $\mathfrak{F}_\infty$ is a σ-field generated by invariant events. This is a general version of Law of Large Numbers.

Lemma 21. *The process $\{X_n\}_{n\geq 0}$ is ergodic.*

Proof. To check if our process $\{X_n\}_{n\geq 0}$ is ergodic, it is sufficient to show that it satisfies the condition

$$Cov(X_n, X_{n+m}) = 0 \text{ as } m \to \infty.$$

This follows directly because

$$Cov(X_n, X_{n+m}) = \frac{\sigma^2 \lambda^m}{1-\lambda^2} \longrightarrow 0$$

as $m \to \infty$ and $\lambda^2 < 1$. □

5.2.2 Applying our CLT

In the previous section, we have shown that the process $\{X_n\}_{n\geq 1}$ is a stationary and ergodic auto-regressive model. A stationary ergodic process is a stochastic process which exhibits both stationarity and ergodicity. In essence, this implies that the random process will not change its statistical properties with time and that its statistical properties (such as the theoretical mean and variance of the process) can be deduced from a single, sufficiently long sample of the process. An ergodic process is one which conforms to the ergodic theorem. The theorem allows the time average of a conforming process to equal the ensemble average, this can be considered as a generalization of the Law of Large Numbers. In practice this means that statistical sampling can be performed at one instant across a group of identical processes or sampled over time on a single process with no change in the measured result.

In this sub-section, we will apply our CLT to an approppriately chosen functional F_n.

For every $n \geq 0$, define

$$\widehat{X}_n = X_n\left(\sqrt{1-\lambda^2}\right).$$

The process $\{\widehat{X_n}\}_{n\geq 0}$ is a multiple of the process $\{X_n\}_{n\geq 1}$, and hence, the former is also stationary and ergodic. As in our proof of the classical Central Limit Theorem, we choose an appropriate sequence of functionals $F_n \in \mathscr{H}_k$ that will satisfy two things:

1. $\mathbb{E}({F_n}^2) \longrightarrow \sigma^2$ as $n \to \infty$

2. $\|DF_n\|_2^2 \longrightarrow k\sigma^2$ in L^2 as $n \to \infty$

When these are satisfied, our CLT says that $F_n \to N(0,1)$.

Define F_n as follows:

$$\begin{aligned} F_n &= \frac{1}{\sqrt{n}}(h_k(\widehat{X}_0) + h_k(\widehat{X}_1) + \cdots + h_k(\widehat{X}_n)) \\ &= \frac{1}{\sqrt{n}}\sum_{i=0}^{n} h_k(\widehat{X}_i), \end{aligned}$$

where h_k is a fixed Hermite polynomial.

Now, we need to verify the assumptions in our CLT - these will be presented as lemmas. The first of which verifies that $F_n \in \mathscr{H}_k$ for every $n \geq 0$; to show this, we will use the following result that is proved in [5]:

If $X \in \mathscr{H}_1$ and $\mathbb{E}\left(X^2\right) = 1$ then $h_k(X) \in \mathscr{H}_k$

Recall that we have computed $\mathscr{H}_1$ in (4.1):

$$\mathscr{H}_1 = \bar{\text{span}}\{\xi_1, \xi_2, \ldots, \xi_N, \ldots\}$$

Lemma 22. *For a fixed $k \geq 2$, $F_n \in \mathscr{H}_k$ for every $n \geq 0$.*

Proof. From the iteration process, we see that

$$\widehat{X}_i = X_i\sqrt{1-\lambda^2} = \sum_{j=0}^{i} c_{i-j}Z_j\sqrt{1-\lambda^2}$$

which implies that $\widehat{X}_i$ is a linear combination of $\xi_0, \xi_1, \ldots, \xi_n, \ldots$, then $\widehat{X}_i \in \mathscr{H}_1$. Next, we show that $\mathbb{E}\left(\widehat{X}_i^2\right)$:

$$\mathbb{E}\left(\widehat{X}_i^2\right) = \mathbb{E}\left(\left(1-\lambda^2\right)X_i^2\right) = \left(1-\lambda^2\right)\mathbb{E}\left(X_i^2\right) = \left(1-\lambda^2\right)\cdot\frac{1}{1-\lambda^2} = 1$$

Hence, by the cited result, each $h_k(\widehat{X}) \in \mathscr{H}_k$ so that by definition of F_n, F_n is also in $\mathscr{H}_k$ for every $n \geq 0$. $\square$

Lemma 23. $\mathbb{E}\left({F_n}^2\right) \longrightarrow \sigma^2$ *as $n \to \infty$.*

Proof. By Theorem (3), we have

$$
\begin{aligned}
\mathbb{E}\left(F_n{}^2\right) &= \frac{1}{n}\sum_{i,p=0}^{n}\mathbb{E}\left(h_k(\widehat{X}_i)h_k(\widehat{X}_p)\right)\\
&= \frac{1}{n}\sum_{i,p=0}^{n}\frac{1}{k!}\left(\mathbb{E}\left(\widehat{X}_i\widehat{X_p}\right)\right)^k\\
&= \frac{1}{n}\sum_{i,p=0}^{n}\frac{1}{k!}\left(\left(1-\lambda^2\right)\mathbb{E}\left(X_iX_p\right)\right)^k\\
&= \frac{1}{n}\sum_{i,p=0}^{n}\frac{1}{k!}\left(\left(1-\lambda^2\right)\frac{\lambda^{|p-i|}}{1-\lambda^2}\right)^k\\
&= \frac{1}{n}\sum_{i,p=0}^{n}\frac{1}{k!}\lambda^{k|p-i|}\\
&= \frac{1}{n}\frac{1}{k!}\sum_{i,p=0}^{n}\lambda^{k|\eta|}\\
&= \frac{1}{n}\frac{1}{k!}\sum_{i=0}^{n}\sum_{\eta=-i}^{n-i}\lambda^{k|\eta|}\\
&= \frac{1}{n}\frac{1}{k!}\sum_{i=0}^{n}\sum_{\eta=-\infty}^{+\infty}\lambda^{k|\eta|}-\ R_n
\end{aligned}
$$

where R_n is equal to

$$
\frac{1}{n}\frac{1}{k!}\sum_{i=0}^{n}\sum_{\eta=-\infty}^{-i-1}\lambda^{k|\eta|}+\frac{1}{n}\frac{1}{k!}\sum_{i=0}^{n}\sum_{\eta=n-i+1}^{+\infty}\lambda^{k|\eta|}.
$$

Note that $R_n \longrightarrow 0$ as $n \to \infty$. To see this, we can re-write R_n as:

$$
R_n = \frac{1}{k!}\left(\frac{1}{n}\sum_{i=0}^{n}a_i+\frac{1}{n}\sum_{i=0}^{n}b_i\right)
$$

where

$$
a_i := \sum_{\eta=-\infty}^{-i-1}\lambda^{k|\eta|} \text{ and } b_i := \sum_{\eta=n-i+1}^{+\infty}\lambda^{k|\eta|}.
$$

Note that a_i is a convergent geometric series because $|\lambda| < 1$ and that its average

$$\frac{1}{n}\sum_{i=0}^{n} a_i \longrightarrow 0 \text{ as } n \to \infty.$$

The same is true for the series b_i. Continuing,

$$\begin{aligned}
\mathbb{E}\left(F_n{}^2\right) &= \frac{1}{n}\frac{1}{k!}\sum_{i=0}^{n}\sum_{\eta=-\infty}^{+\infty} \lambda^{k|\eta|} \\
&= \frac{1}{k!}\left[\sum_{\eta=-\infty}^{-1} \lambda^{k|\eta|} + \lambda^{k\cdot 0} + \sum_{\eta=1}^{+\infty} \lambda^{k|\eta|}\right] \\
&= \frac{1}{k!}\left[1 + 2\sum_{\eta=1}^{+\infty} \lambda^{k|\eta|}\right] \\
&= \frac{1}{k!}\left[1 + 2\left(\frac{\lambda^k}{1-\lambda^k}\right)\right] \\
&= \frac{1}{k!}\left[\frac{1+\lambda^k}{1-\lambda^k}\right] \\
&= \sigma^2
\end{aligned}$$

This proves that $\mathbb{E}\left(F_n{}^2\right) = \sigma^2$ as $n \to \infty$. □

The next job is to look at the square of the 2-norm of the derivative of DF_n:

Lemma 24. *$\|DF_n\|_2^2 \longrightarrow k\sigma^2$ in L^2 as $n \to \infty$.*

For $l = 0, 1, 2, \ldots$ the components of the derivative of DF_n are

$$D^l F_n = \begin{cases} \frac{1}{\sqrt{n}} \sum_{i=0}^{n} h_k'(\widehat{X}_i) c_i & l = 0 \\ \frac{1}{\sqrt{n}} \sum_{i=0}^{n} h_k'(X_i) \cdot c_{i-l} \cdot \sqrt{1-\lambda^2} & l \geq 1,\ c_{i-l} = 0, l > i, l \leqslant i \leqslant n \end{cases}$$

Since

$$DF_n = (D^0 F_n, D^1 F_n, \ldots, D^n F_n, \cdots)$$

so the square of its 2-norm is

$$
\begin{aligned}
\|DF_n\|_2^2 &= \sum_{l=0}^{n}(D^l F_n)^2 \\
&= \sum_{l=1}^{n}\left(\frac{1}{\sqrt{n}}\sum_{i=0}^{n}h_k'(\widehat{X}_i)\cdot c_{i-l}\right)^2\left(1+\lambda^2\right)+\left(\frac{1}{\sqrt{n}}\sum_{i=0}^{n}h_k'(\widehat{X}_i)c_i\right)^2 \\
&= \frac{1}{n}\sum_{l=1}^{n}\sum_{i,p=0}^{n}h_k'(\widehat{X}_i)\cdot c_{i-l}\cdot h_k'(\widehat{X}_p)\cdot c_{p-l}\cdot\left(1-\lambda^2\right)+\frac{1}{n}\sum_{i,p=0}^{n}h_k'(\widehat{X}_i)h_k'(\widehat{X}_p)c_ic_p \\
&= \frac{1}{n}\sum_{i,p=0}^{n}h_k'(\widehat{X}_i)h_k'(\widehat{X}_p)\left[\sum_{l=1}^{min(i,p)}c_{i-l}\cdot c_{p-l}\cdot\left(1-\lambda^2\right)+c_ic_p\right]
\end{aligned}
$$

We apply a change of variable by setting $\eta = p-i$, so we have

$$
\begin{aligned}
\|DF_n\|_2^2 &= \frac{1}{n}\sum_{i,p=0}^{n}h_k'(\widehat{X}_i)h_k'(\widehat{X}_{i+\eta})\cdot K(\eta)\cdot\left(1-\lambda^2\right) \\
&= \frac{1}{n}\sum_{i=0}^{n}\sum_{\eta=-i}^{n-i}h_k'(\widehat{X}_i)h_k'(\widehat{X}_{i+\eta})\cdot K(\eta)\cdot\left(1-\lambda^2\right) \\
&= \frac{1}{n}\sum_{i,p=0}^{n}h_k'\left(\widehat{X}_i\right)h_k'\left(\widehat{X}_p\right)\lambda^{|p-i|}\left(1-\lambda^2\right) \\
&= \frac{1}{n}\sum_{i=0}^{n}\sum_{\eta=-i}^{n-i}h_k'\left(\widehat{X}_i\right)h_k'\left(\widehat{X}_{i+\eta}\right)\lambda^{|\eta|}\left(1-\lambda^2\right) \\
&= \left(1-\lambda^2\right)\frac{1}{n}\sum_{i=0}^{n}\sum_{\eta=-\infty}^{+\infty}h_k'\left(\widehat{X}_i\right)h_k'\left(\widehat{X}_{i+\eta}\right)\lambda^{|\eta|}-\ R_n
\end{aligned}
$$

where R_n is composed of

$$
\frac{1}{n}\sum_{i=0}^{n}\sum_{\eta=-\infty}^{-i-1}h_k'\left(\widehat{X}_i\right)h_k'\left(\widehat{X}_{i+\eta}\right)\lambda^{|\eta|}\left(1-\lambda^2\right)
$$

and

$$
\frac{1}{n}\sum_{i=0}^{n}\sum_{\eta=n-i+1}^{+\infty}h_k'\left(\widehat{X}_i\right)h_k'\left(\widehat{X}_{i+\eta}\right)\lambda^{|\eta|}\left(1-\lambda^2\right).
$$

It can be shown that each of these approaches 0 as $n \to \infty$. For $i = 0, 1, ..., n$, define

$$Y_i = \sum_{\eta=-\infty}^{+\infty} h_k'\left(\widehat{X}_i\right) h_k'\left(\widehat{X}_{i+\eta}\right) \lambda^{|\eta|}.$$

Let us show that Y_i is well-defined, stationary, and ergodic.

Claim 1: Y_i is well-defined.

Proof.

$$\begin{aligned} Y_i &= \sum_{\eta=-\infty}^{+\infty} h_k'\left(\widehat{X}_i\right) h_k'\left(\widehat{X}_{i+\eta}\right) \lambda^{|\eta|} \\ &= h_k'\left(\widehat{X}_i\right) \sum_{\eta=-\infty}^{+\infty} h_k'\left(\widehat{X}_{i+\eta}\right) \lambda^{|\eta|}. \end{aligned}$$

We need to show that the series $\sum_{\eta=-\infty}^{+\infty} h_k'\left(\widehat{X}_{i+\eta}\right) \lambda^{|\eta|}$ is convergent. This is true because its norm $\sum_{\eta=-\infty}^{+\infty} \left\|h_k'\left(\widehat{X}_{i+\eta}\right)\right\|_2 \left|\lambda^{|\eta|}\right| < \infty$. □

Claim 2: Y_i is strictly stationary.

Proof. Strictly stationary means that the process $(Y_0, Y_1, \dots Y_n)$ will have the same distribution as the process $(Y_m, Y_{1+m}, \dots Y_{n+m})$ for some m that is,

$$(Y_0, Y_1, \dots Y_n) \Leftrightarrow (Y_m, Y_{1+m}, \dots Y_{n+m}).$$

This can be seen from

$$Y_i = h_k'\left(\widehat{X}_i\right) \sum_{\eta=-\infty}^{+\infty} h_k'\left(\widehat{X}_{i+\eta}\right) K(\eta) \text{ for } (i = 0, 1, \dots, n)$$

and

$$Y_{i+m} = h_k'\left(\widehat{X}_{i+m}\right) \sum_{\eta=-\infty}^{+\infty} h_k'\left(\widehat{X}_{i+m+\eta}\right) K(\eta) \text{ for } (i = 0, 1, \dots, n)$$

Since $\widehat{X}_i$ is also strictly stationary and Gaussian which implies that

$$\left(\widehat{X}_i\right)_{i=-\infty}^{+\infty} = \left(\widehat{X}_{i+m}\right)_{i=-\infty}^{+\infty}$$

This proves that Y_i is strictly stationary. $\square$

Claim 3: Y_i is ergodic.

Proof. Ergodicity of Y_i follows from the fact that the process $\{X_k\}_k$ is ergodic. This follows from the implication that says if X_i is ergodic then Y_i is ergodic (see [2]). $\square$

Since Y_i is ergodic, then by ergodic theorem (see [5]), we have,

$$\begin{aligned}
\|DF_n\|_2^2 &= \left(1-\lambda^2\right)\frac{1}{n}\sum_{i=0}^{n}\sum_{\eta=-\infty}^{+\infty} h_k'\left(\widehat{X}_i\right) h_k'\left(\widehat{X}_{i+\eta}\right)\lambda^{|\eta|} \\
&\longrightarrow \mathbb{E}\left(\sum_{\eta=-\infty}^{+\infty} h_{k-1}\left(\widehat{X}_i\right) h_{k-1}\left(\widehat{X}_{i+\eta}\right)\lambda^{|\eta|}\right) \\
&= k\sigma^2.
\end{aligned}$$

This proves our third lemma for this sub-section. Thus, by our CLT, we have that

$$F_n \to N(0,1) \text{ as } n \to \infty \text{ in distribution.}$$

Bibliography

[1] V. Bogachev, *Gaussian Measures*, Mathematical Surveys and Monographs, Volume 62, American Mathematical Society, 1998. Cited on

[2] R. Durrett, *Probability: Theory and Examples, 3rd Edition*, Brooks/Cole, USA, 2005. Cited on 61

[3] K. Hoffman and R. Kunze, *Linear Algebra, 2nd Edition*, Prentice Hall, USA, 1961. Cited on

[4] J. Jacod and P. Protter, *Probability Essentials, 2nd Edition*, Springer-Verlag, Berlin Heidelberg, 2003. Cited on 7

[5] D. Nualart, *The Malliavin calculus and related topics 2nd Edition*, Springer-Verlag, Berlin Heidelber, 2006. Cited on 13, 17, 20, 26, 37, 55, 61

[6] D. Nualart and G. Peccati, *Central limit theorems for sequences of multiple stochastic integrals*, Volume 33, Number 1, 177-193, 2005. Cited on 8, 26

[7] S. Ross, *Stochastic Processes, Second Edition*, John Wiley and Sons, 1996, USA. Cited on 50

[8] S. Ross, *Introduction to Probability Models, Ninth Edition*, Elsevier, 2007, USA. Cited on 50

[9] W. Rudin, *Principles of Mathematical Analysis, Third Edition*, Mc-Graw Hill, 1976. Cited on

[10] F. Max Stein, *Orthogonal functions whose kth derivatives are also orthogonal*, SIAM Review, Vol. 1, No. 2, pp. 167-170, 1959.

Cited on 12